

**2019 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test**

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. For all real values of x , $6x^2 + bx + c = 3(ax^2 + 3x + 4)$. Determine the sum $(a + b + c)$.

2. The sum of the mode and median is k larger than the arithmetic mean for the set of numbers $\{2, 2, 8, 2, 0, 1, 5\}$. Determine the value of k . Express your answer as a common or improper fraction reduced to lowest terms.

3. A triangle with sides extended is shown (but not necessarily drawn to scale). It



has angle measures as labeled. $a : b : c = 5 : 7 : 8$. Determine the ratio of $x : y : z$. Express your answer in the form $x : y : z$.

4. A prime number less than 20 is randomly selected. Determine the probability that the number selected is even. Express your answer as a common fraction.

5. x and 30.

9. x is a positive odd integer. Determine the sum of all the distinct value(s) of x such that $-(x-11) = |x-11|$

10. A triangular piece broke off rectangle $ABDF$ leaving trapezoid $ACDF$. $BD = 16$, $BC = 7$, $FD = 24$, and E is the midpoint of \overline{FD} . Find the perimeter of $\triangle ACE$.



Name:

Team Code:

**2019 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test**

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form.

Exact answers are to be given unless otherwise indicated. Round answers to the nearest hundredth.

1.

11.

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Must be this exact decimal.

3. **15:13:12** Must be exactly this format. **13.**

Must be this reduced fraction.

171 (feet optional)

5. **15.** $\frac{40}{49}$ Must be this reduced fraction.

6. **13,680** (people optional) **16.** **75**

7. $\frac{169}{12}$ Must be this improper fraction. **17.** $\frac{1}{2}$ Must be this reduced fraction.

8. **19** **18.** **17**

9. **36** **19.** **2.580** Must be this exact decimal (trailing zero is necessary).

10. **60** **20.** **(6,8)** Must be this ordered pair.

2019 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem is worth 10 points.

3. Determine the number of distinct prime factors for the number 426,888.
4. Determine the number of rectangles in the figure on the right.
5. Jordan drove to a town 25 miles away at an average speed of 50 miles per hour. The return trip along the same route took 20 minutes longer than the trip to town. Determine the average speed in miles per hour for Jordan's round trip. Express your answer as an exact decimal.



7. Determine the exact distance from the center of the circle given by $x^2 + 8x + y^2 - 6y + 3 = 2$ to the

10. Suppose $x = \sqrt{20} + \sqrt{20} + \sqrt{20} + \dots$. Determine the exact value of x .

Name:

Team Code:

**2019 John O'Bryan Mathematical Competition
Junior/Senior Individual Test**

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

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Name: _____ **ANSWERS** _____

Team Code: _____

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- | | | | | | |
|-----|----------------|---------------------------------|-----|---------------|--------------------------------|
| 1. | 4 | | 11. | $\frac{1}{2}$ | Must be this reduced fraction. |
| 2. | 15 | | 12. | (3,1) | Must be this ordered pair. |
| 3. | 4 | | 13. | $10\sqrt{59}$ | Must be this simplified form. |
| 4. | 100 | | 14. | $\frac{6}{5}$ | Must be this reduced fraction. |
| 5. | 37.5 | (miles per hour optional units) | 15. | 3 | |
| 6. | $\frac{5}{6}$ | Must be this reduced fraction. | 16. | 58.73 | Must be exactly this answer. |
| 7. | $2\sqrt{5}$ | Must be this simplified form. | 17. | 2.58 | Must be exactly this answer. |
| 8. | 885.8 | Must be exactly this answer. | 18. | 144 | |
| 9. | $\frac{10}{3}$ | Must be this simplified form. | 19. | 48 | |
| 10. | 5 | | 20. | 7 | |

2019 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

*****Calculators may not be used on the first four questions*****

1. Let $k = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}$ with $a + b + c = 15$ and $abc = 5$. Let $w = \frac{z}{xy} + \frac{y}{xz} + \frac{x}{yz}$ with $x^2 + y^2 + z^2 = 12$ and $xyz = 3$. Determine $(k + w)$.
2. Let $f(x) = \sin^2(19x) + \cos^2(19x)$ and $g(x) = \begin{vmatrix} 2 & x \\ -1 & 9 \end{vmatrix}$. If $k = f(g(2))$ and $w = g(f(2))$, then determine $(k + w)$.
3. Let $\log_8 64 + \log_2 k - 2 \log_4 8 = 4$ and $3 \sin \frac{\pi}{6} + 4 \tan \theta\pi + 7 \cos \frac{5\pi}{3} = 1$ with $0 \leq \theta \leq 1$. Determine $\frac{k}{\theta}$.
4. Let s be the coefficient of x^4 in $(x + 1)^7$ and t be the coefficient of y^3 in $(y + 2)^5$. Determine $(s - t)$.

*****Calculators may be used on the remaining questions*****

5. Let a be a positive integer number base so that $28_a = 132_5$. Let $3^{b-2c} = \frac{3^{2b-c}}{81}$. Determine $(a + b + c)$.

6. A circle C has a diameter with endpoints $A(4, -1)$ and $B(2, 5)$. If C has center (h, k) and radius

Names: _____

School: _____

2019 John O'Bryan Mathematical Competition

Answers for the Two-Person Sneed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless

1.

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4.

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SCORE

T1.

T2.

3.

24

-5

2019 John O'Bryan Mathematics Competition :: 5-Person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper.

Questions will not be scored without the following two things:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. Teams must **show complete solutions (not just answers)** to receive

credit. More specific instructions are read verbally at the beginning of the test.

2019 John O'Bryan Mathematics Competition :: 5-Person Team Test

1. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_6x^6$ be an arbitrary polynomial of degree 6. Define

$$g(x) = f(x) - f(1/x).$$

(a) Find $g(x)$ and $g(1/x)$. Note: your answer may include the constant coefficients a_0, a_1

2. A non-negative integer n is a *square number* if $n = k^2$ for some integer k .

(a) Determine how many positive integers n with $n \leq 20$ can be written as the sum of two square numbers (not necessarily distinct). That is, $n = a^2 + b^2$ for some integers a and b .

Solution: Note that 0, 1, 4, 9, and 16 are the squares that are at most 20. There are $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$ possible sums of distinct squares (only one of which is more than 20), and 5 possible sums of the same square (two of which are not in the interval $[1, 20]$). This gives a maximum of $10 + 5 = 15$ sums of two squares. Listing the possible positive sums to check for repeats gives

$$1, 4, 9, 16, 1, 4, 9, 16, 1, 4, 9, 16, 1, 4, 9, 16.$$

Therefore, there are 12 such positive integers.

(b) Assume $n = a^2 + b^2 + c^2$ with $n > 0$. Note that $a^2 + b^2 - c^2 > 0$. Prove that n is the sum of three, positive square numbers.

Solution: Let $n = a^2 + b^2 + c^2$. Expanding and rearranging gives

$$\begin{aligned} (a^2 + b^2 + c^2)^2 &= a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \\ &= (a^4 + b^4 + c^4 + 2a^2b^2 - 2b^2c^2 - 2c^2a^2) + 2a^2b^2 + 2b^2c^2 \\ &= (a^2 + b^2 - c^2)^2 + (c)^2 + (c)^2, \end{aligned}$$

which is the sum of three squares.

3 Let $s(n)$ be the function that sums the digits of a positive integer n ; e.g. $s(123) = 1 + 2 + 3 = 6$.

(a) Determine $s(n) + s(s(n))$ when $n = 10^k$.

Solution: Note $s(10^k) = 1 + 0 + \dots + 0 = 1$. Then $s(s(10^k)) = s(1) = 1$.
 $s(n) + s(s(n)) = 1 + 1 = 2$.

(b) Consider M where $M = s(n) + s(s(n))$ for some positive integer n . Prove that M is a multiple of an integer K with $K > 1$.

Solution: Note that a positive integer is divisible by 3 exactly when the sum of its digits is divisible by 3. Thus $s(n) \equiv n \pmod{3}$. Hence, $s(n) + s(s(n)) \equiv n + s(n) \equiv n + n \equiv 2n \pmod{3}$. Therefore, M must be a multiple of 3.

4. Let A, B, C be vertices of a triangle with angles a, b, c (see Figure 1). Let I be the center of the circle outside of $\triangle ABC$ tangent to \overline{BC} , \overline{AB} , and \overline{AC} . Note that I is the intersection of the bisector of the angle A , and the bisector of the exterior angle at vertex B .

(a) Determine $\angle AIB$ in terms of a .

Solution: Notice that $\angle IBA = \frac{a}{2}$, $\angle ICB = \frac{180-b}{2}$ and $\angle AIB = \angle ICB + \angle IBA = 90 + \frac{b}{2}$.
 From the angle sum for $\triangle AIB$ and $\triangle ABC$ we have

$$\angle AIB = 90 + \frac{b}{2}$$

5. At the start of the school year Jamie has 18 classmates in maths class, for a total of 19 students in the class. Each of Jamie's 18 classmates has a unique number of friends in the class (which could include Jamie). For example, Alex and Riley can **not**

6. Consider the operation of *square-replacement*, in which a square is replaced with four equal-sized squares (i.e. splitting the square into quarters).

(a) Starting from the unit square, consider the iterative process of square-replacing every square. Let $\alpha(k)$ denote the sum of perimeters for *all* squares after k iterations. Determine $\alpha(k)$ for $k=0, 1, 2, 3$, and general k .

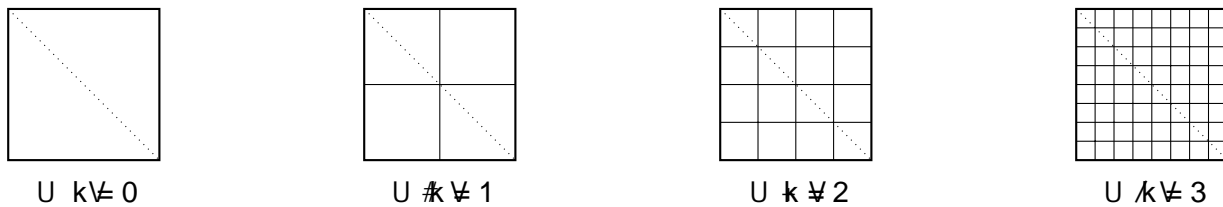


Figure 2: Iterations of square replacement, having 1, 4, 16, and 64 squares, respectively.

Solution: After k iterations, there are $(2^k)^2$ squares each with side length 2^{-k} . Thus

$$\alpha(k) = 2^k \cdot (2^k - 1) \cdot 2^{-k+1}.$$

So $\alpha(0) = 4$, $\alpha(1) = 12$, $\alpha(2) = 20$, and $\alpha(3) = 28$.

(b) Continuing from part (a), let $\delta(k)$ denote the sum of perimeters of squares that overlap the main diagonal of the unit square (possibly at a corner) after k iterations. Determine $\delta(k)$ for $k=0, 1, 2, 3$, and general k .

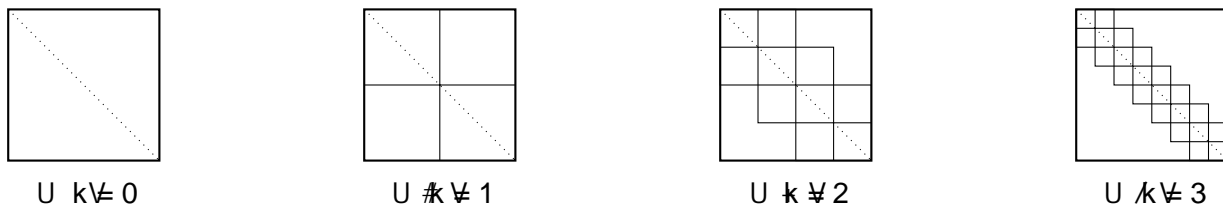


Figure 3: Squares contributing to $\delta(k)$, having 1, 4, 10, and 22 squares, respectively.

Solution: After k -iterations, there are $(2^k)^2$ squares whose diagonal lies on the main diagonal of the unit square and $(2^k - 1)$ squares that overlap the main diagonal only at a corner. The side lengths of these squares is still 2^{-k} . Thus

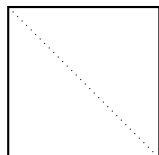
$$\delta(k) = (2^k - 1) \cdot (2^k + (2^k - 1)) \cdot 2^{-k+1}.$$

So $\delta(0) = 4$, $\delta(1) = 10$, $\delta(2) = 18$, and $\delta(3) = 28$.

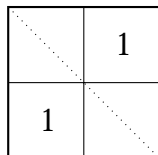
(c) Is it possible to square-replace in a way that $\delta(k) > 4$ for some k ?

Solution: Yes. Note that $\delta(k) < 4$ for square-replacing every square, which does not work. Square-replacing the unit square gives four squares. We call the two squares with sides $\frac{1}{2}$ and overlapping the main diagonal only at a corner level-1 squares, and then square-replace the remaining squares. Next, we call the four squares with sides $\frac{1}{4}$ and overlapping the main diagonal only at a corner level-2 squares, and then square-replace the remaining squares

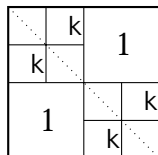
(leaving the level-1 squares in tact). We continue in this way until we have 505 levels of squares. There are 2^{n-1} squares at level n , each with side 2^{-n} . Thus, the total perimeter of all level- n squares is $4 \cdot 2^{n-1} \cdot 2^{-n} = 2$, and the total perimeter of all squares overlapping the diagonal is $2 \cdot 2^{n-1} = 2^n$, as desired.



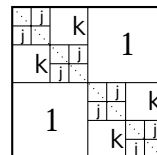
$U_{k \neq 0}$



$U_{k \neq 1}$



$U_{k \neq 2}$



$U_{k \neq 3}$