2020 John O'Bryan Mathematics Competition :: 5-Person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper

[redacted]

4. Let $a \neq -1$.

   (a) Calculate $\dfrac{a^5+1}{a+1}$.

   (b) Assume $a+1$ is divisible by 5. Show that $a^4 - a^3 + a^2 - a + 1$ is divisible by 5.

   (c) Let $k \geq 1$ and assume $4^{5^{k-1}} + 1$ is divisible by 5^k. Show that $4^{5^k} + 1$ is divisible by 5^{k+1}

5. In discrete time steps (i.e. $t = 0, 1, 2, 3, 4, \ldots$), a contagion spreads around vertices in a graph where infected vertices are solid and healthy vertices are open. If an infected vertex has only one healthy neighbor, that healthy neighbor becomes (and stays) infected at the next time step. Starting (below

(a) Determine the maximum rook social distance on the chess boards below.

the entire graph as follows: $A \to C$ and $B \to D$; then $C \to E$; then E F; and finally $D \to G$

(b) How many ways can 3 rooks can be placed on a 3×3 chess board so that the squares they are

[redacted]

Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in

[redacted]

11. Let k be the smallest integer and let w be the largest integer such that both k and w are values for x that satisfy the inequality $|2x-1|<15$. Find the value of $(3k+2w)$.

[redacted]

=   =

A

$z°$

$y°$

$x°$

C   B

15. A circle has an equation of $(x-8)^2 + y^2 + 10y = -13$. Find the length of the radius of the circle. Give your

[redacted]

answer in the form $a\sqrt{b}$ where both a and b are integers and b is prime.

[redacted]

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Name: Team Code:

2020 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

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2020 John O'Bryan Mathematical Competition

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise

8. Find the remainder when $(11)^{(27)^6}$ is divided by 7.

9. The terms of an arithmetic sequence are: 50, 75, 100, ... The terms of the triangular sequence whose n^{th}

11 Given the set: $\{\log_2(16), \log_3(16), \log_4(16), \log_5(16), \log_6(16), \log_7(16), \log_8(16)\}$. If one of the members of the set is drawn at random, find the probability that the member drawn could represent a positive integer. Express your answer as a common fraction reduced to lowest terms.

12. Find the ordered pair that represents the sum of the following two vectors: $(-5,6)$ and $(17,7)$

13. In ΔABC, find the exact length of $\overline{AC}$ if AB=20, BC=80, and $\angle ABC = 120°$. Give your answer in the form $a\sqrt{b}$ where both a and b are integers and a is as large as possible.

14. [redacted]

Name: ______________________ Team Code: ________

2020 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1.	**11.**
2.	**12.**
3.	**13.**
4.	**14.**
5.	**15.**
6.	**16.**
7.	**17.**
8.	**18.**
9.	**19.**
10.	**20.**

2020 John O'Bryan Mathematics Competition :: 5-Person Team Test

1. Aliyah and Ynez both have sticks (not necessarily the same size). First, Aliyah breaks her stick into [redacted]

[redacted]

2. Assume x, y, z are integers with $x \geq y > 0 > z$. Consider the equation

$$x^2 + y^2 + z^2 = x^3 + y^3 + z^3.$$

(a) Find a solution where $|x| = |z|$ and $|x| + |y| + |z| < 10$.

Solution: Since $|x| = |z|$ and $x > 0 > z$, we have $z = -x$, which gives

$$2x^2 + y^2 = y^3.$$

Solving for x gives $x = \sqrt{\frac{y^3 - y^2}{2}} = y\sqrt{\frac{y-1}{2}}$. Since x is an integer, $\frac{y-1}{2}$ must be a perfect square. Since $x + y + |z| < 10$ and $x \geq y$, $\frac{y-1}{2} = 1$. Therefore, $y = 3$, which makes $x = 3$ and $z = -3$.

(b) Find a solution where $y = \frac{x}{2}$.

Solution: To construct an example, when $y = \frac{x}{2}$, we have

$$\begin{aligned} 0 &= x^3 + y^3 + z^3 - x^2 - y^2 - z^2 \\ &= \frac{9x^3}{8} + z^3 - \frac{5x^2}{4} - z^2. \end{aligned}$$

To eliminate the z terms, pick $z = -x$ so that

$$\begin{aligned} 0 &= \frac{x^3}{8} - \frac{9x^2}{4} \\ &= \frac{x^2}{4}\left(\frac{x}{2} - 9\right), \end{aligned}$$

which implies $x = 18$, making $y = 9$ and $z = -18$.

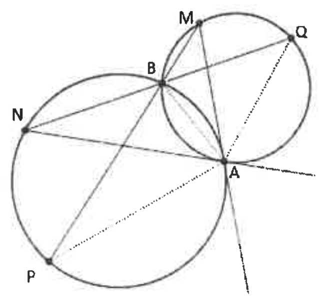
(c) Show that there are infinitely many solutions to the equation.

Solution: When $z = -x$, the analysis in part (a) gives that the equation is satisfied whenever

[redacted]

3. In the circles below, note that: (1) the circles intersect at A and B; (2) $\overrightarrow{MA}$ is tangent to the circle containing P; and (3) $\overrightarrow{NA}$ is tangent to the circle containing Q.

Recall that the *Inscribed Angle Theorem* states that an inscribed angle is half the measure of the arc it intercepts (i.e. subtends). As a consequence, each of the two adjacent angles formed by a tangent and a chord drawn from the point of tangency is equal to half the measure of the arc it intercepts (i.e. subtends).



(a) Prove that triangles AQN and AMP are similar.

Solution: By the Inscribed Angle Theorem, $\angle P = \angle N$ and $\angle Q = \angle M$. Combined with the angle sum formula, these imply that $\angle NAQ = \angle PAM$. Since both triangles have same angle measures, they are similar.

[redacted]

4. Let $a \neq -1$.

(a) Calculate $\dfrac{a^5+1}{a+1}$.

Solution: Note that $a^5 + 1 = (a+1)(a^4 - a^3 + a^2 - a + 1)$.

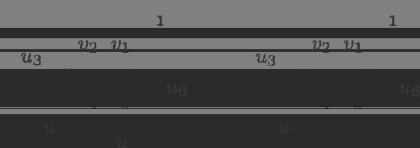
(b) Assume $a+1$ is divisible by 5. Show that $a^4 - a^3 + a^2 - a + 1$ is divisible by 5.

Solution: Since 5 divides $a+1$, $a+1 = 5m$ for some $m \in \mathbb{Z}$. Thus $a = 5m - 1$ As such

$$\begin{aligned} a^4 - a^3 + a^2 - a + 1 &= (5m-1)^4 - (5m-1)^3 + (5m-1)^2 - (5m-1) + 1 \\ &= (5k_4 + 1) - (5k_3 - 1) + (5k_2 + 1) - (5m - 1) + 1 \\ &= 5(k_4 - k_3 + k_2 - m) + 5 \\ &= 5(k_4 - k_3 + k_2 - m + 1), \end{aligned}$$

[redacted]

5. In discrete time steps (i.e. $t = 0, 1, 2, 3, 4, \ldots$), a contagion spreads around vertices in a graph where infected vertices are solid and healthy vertices are open. If an infected vertex has only one healthy



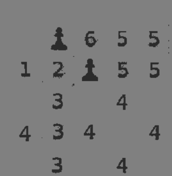
graph, C_n, has n vertices in a circle, each with a single additional neighbor; see examples of C_4, C_5, C_6, and C_7 below.

(a) If u_1, u_4 are initially infected, how long does it take to completely infect C_4?

In general, C_{4m} needs $2m$ initial and $m+1$ steps; C_{4m+1} needs $2m+1$ initial and $m+2$ steps;
[illegible]

6. In chess, a rook (i.e. castle) can move any number of spaces vertically or horizontally. We define the *rook social distance* between squares on a chess board as the number of moves it takes a rook to move

[redacted]



Suppose there are only 2 pawns. If they are in the same row, then the maximum rook social

[redacted]

Name: ______ANSWERS______ Team Code: _______

2020 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 1507
2. 500 — Percent optional
3. -23
4. 1
5. 2
6. 101
7. 4/7 — Must be this reduced fraction.
8. 1050 — (cents optional)
9. 48
10. 2346
11. 4
12. 144
13. 16
14. 132
15. $2\sqrt{3}$ — Must be this exact answer.
16. 46
17. 50
18. 65
19. $8\sqrt{5}$ — Must be this
20. 25

Awards Lists and Solutions to the Team Competition may be found at
http://math.nku.edu/job

Name: ANSWERS______ **Team Code:**

[redacted]

1. 36

2. $\frac{75}{13}$ Must be this [redacted]

3. 1080

4. 60

5. 22

6. $6\sqrt{3}$ Must be exactly this answer.

7. 4

8. 1

9. 1326

10. 9

11. $\frac{2}{7}$ Must be this reduced fraction.

12. (12,13) Must be this ordered pair.

13. $20\sqrt{21}$ Must be exactly this answer.

14. 88.67 Must be exactly this decimal.

15. $\frac{5}{3}$ Must be this improper fraction.

16. 18

17. Lose 18 Must be exactly this answer.

18. $8\sqrt{5}$ Must be exactly this answer.

19. 144

20. 19