

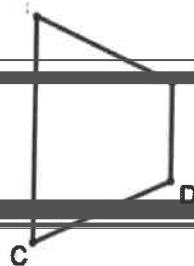
**2023 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test**

Directions: Please answer all questions on the answer sheet.

[The answer sheet area is heavily obscured by a large grey block and contains illegible text.]

13. When $x^4 - 33$ is divided by $x - 3$, find the remainder.

14. If $43^\circ 20'$ were expressed in terms of degrees, the answer would be $43^\circ 1$.



Name: _____

Team Code: _____

**2023 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test**

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem is worth 5 points.

1.

11.

2.

12.

3.

13.

4.

14.

5.

15.

6.

16.

7.

17.

8.

18.

9.

19.

10.

20.

Name: **ANSWERS**

Team Code:

**2023 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test**

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measure.

4. $1/3$ Must be this fraction
6. $171/29$ Must be this fraction
7. $11/18$ Must be this fraction
8. 74
9. 66
10. 405
- 13.
14. $509/120$ Must be this fraction
- 15.
16. 5
17. 4.4 Must be this decimal
18. 314.8 Must be this decimal
19. 73
20. $221/30$ Must be this fraction

2023 John O'Bryan Mathematical Competition

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form.



12. An apartment rental company has 3000 apartments available, and 1900 are presently rented at \$950 per month. The company has decided it will only raise or lower the rent per month on all apartments by integral increments of \$40. A survey has shown that, so long as there are apartments available for each

Name:

Team Code:

2023 John O'Bryen Mathematical Competition

1.

1.

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20.

Name: ANSWERS

Team Code: _____

2023 John O'Bryan Mathematical Competition

2. $13/15$

Must be this fraction.

12. 870

3. 1.2

Must be this decimal.

13. 11

4. $1/6$

Must be this fraction.

14. 32

5. -4.8

Must be this decimal.

15. $32/53$

Must be this fraction.

6. 10

16. 5

7. π

17. 172

8. $1/2$

Must be this fraction.

18. 1073741823

9. 13.13

Must be this decimal.

19. 4.00

10. 17

20. 27719

y

d

Names:

School:

**2023 John O'Bryan Mathematical Competition
Two-Person Speed Event**

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

SCORE

Calculators are not allowed to be used on the first four questions!

1.

2.

This competition consists of several

3.

competitive rounds. Correct answers will receive the following scores:

4.

1st: 7 points
2nd: 5 points
All Others: 3 points

5.

6.

7.

8.

SCORE

T1.

T2.

Names:

School:

2023 John O'Bryan Mathematical Competition

2.

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point value.

3.

4.

-21

5.

63

SCORE

Calculators are not allowed to be used on the first four questions!

6.

3159

7.

6.65

Must be this decimal

This competition consists of eight competitive rounds. Correct answers will receive the following

8.

1

SCORE

T1.

784

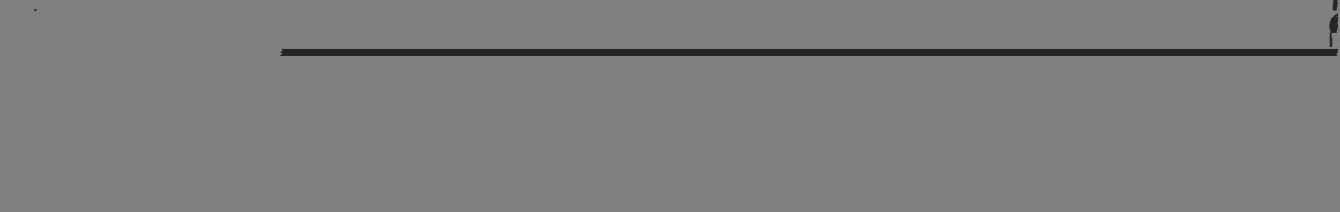
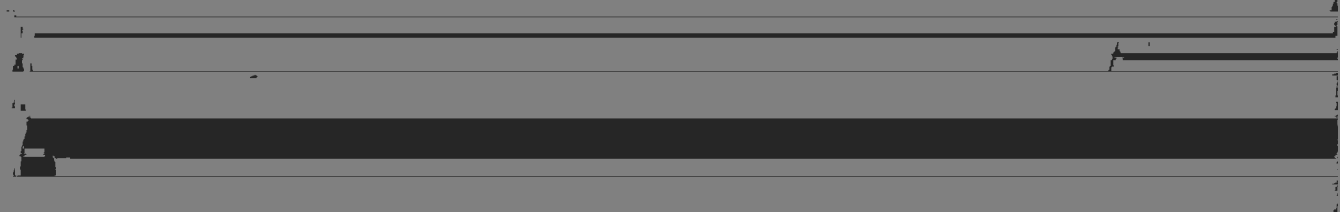
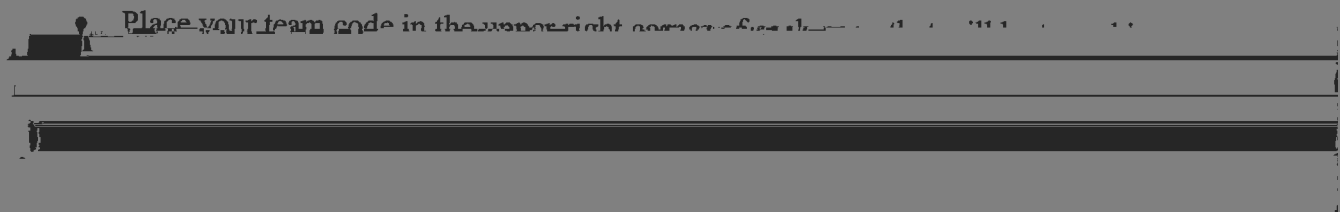
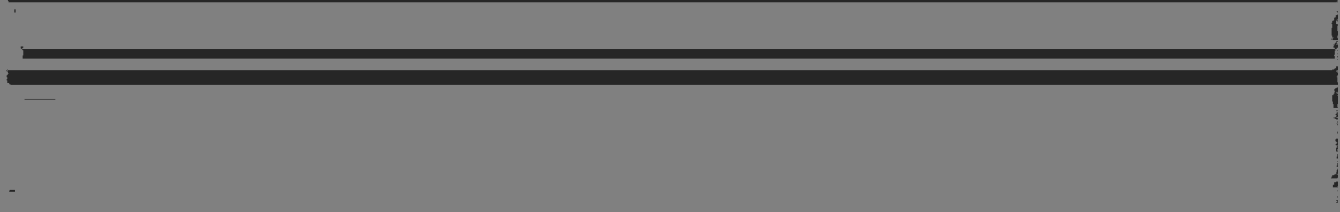
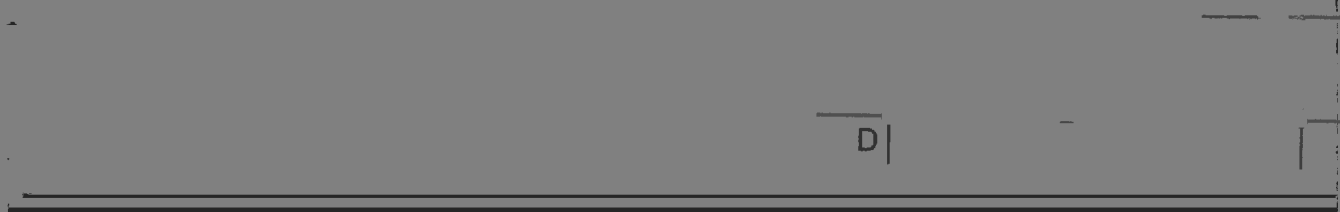
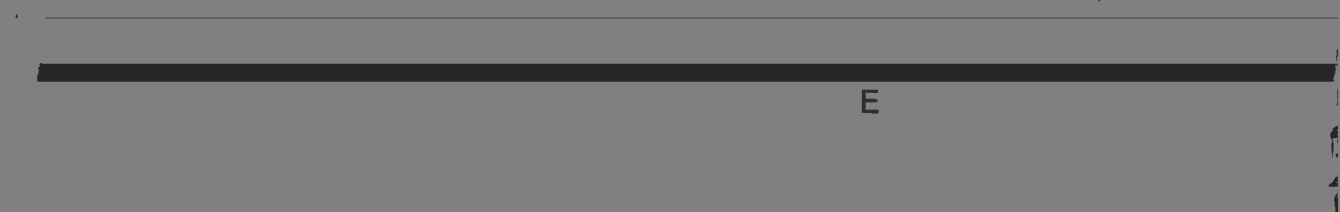
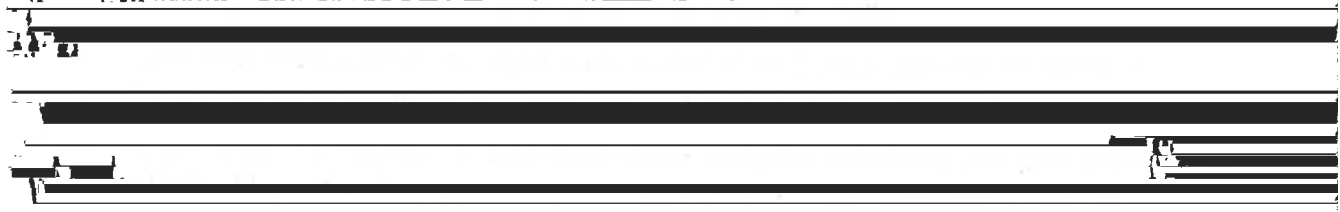
T2.

30.5

Must be this decimal

**2023 John O'Bryan Mathematics Competition
5-Person Team Test**

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper. Questions will not be read.



Place your team code in the upper right corner of this sheet.

4. Complete each of the following. Remember for all problems you must show or explain how you obtained your answer.

a. Determine the positive integer c for which $\frac{1}{4} - \frac{1}{c} = \frac{1}{6}$

b. Determine the number of pairs of positive integers (d, e) for which $\frac{1}{d} - \frac{1}{e} = \frac{1}{12}$

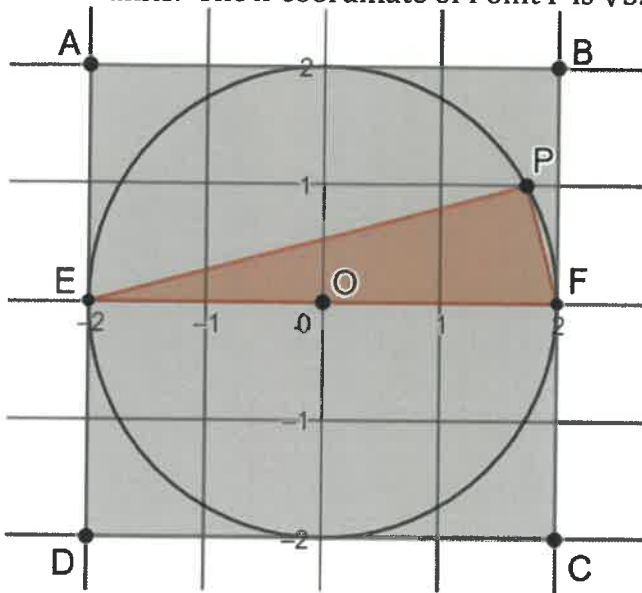
Show that for any positive integers d, e

$$\neq$$

$$=$$

John O'Bryan 2023 Team Test Key

1. In the figure below, quadrilateral ABCD is square and inscribed circle O has a radius of 2 units. The x-coordinate of Point P is $\sqrt{3}$.



a. Find the area of triangle EPF.

The equation of the circle is $x^2 + y^2 = 4$. If the x-coordinate of P is $\sqrt{3}$, then $(\sqrt{3})^2 + y^2 = 4$. So, $y = 1$. With a base of 4 and altitude of 1, the area of triangle EPF is $(1/2)(4)(1) = 2$ square units.

b. Find the sum of the squares of the distances from point P to each vertex of the square.

Using the distance formula:

$$PA^2 = (\sqrt{3} - (-2))^2 + (1 - 2)^2 = 8 + 4\sqrt{3}$$

$$PB^2 = (\sqrt{3} - 2)^2 + (1 - 2)^2 = 8 - 4\sqrt{3}$$

$$PC^2 = (\sqrt{3} - 2)^2 + (1 - (-2))^2 = 16 - 4\sqrt{3}$$

$$PD^2 = (\sqrt{3} - (-2))^2 + (1 - (-2))^2 = 16 + 4\sqrt{3}$$

$$\text{So, } PA^2 + PB^2 + PC^2 + PD^2 = 48$$

- c. Does the answer to part (b) depend upon the location of point P or will you obtain the same sum of squares for any point on the circle? Justify your answer.

No, the sum of the squares remains the same (48) for any point on the circle. To show this is true for any point on the top half of the circle (the case for points on the lower half

A series of horizontal lines for writing an answer, consisting of 15 lines. The lines are evenly spaced and extend across the width of the page.

2. On day 1, Paris writes the number 1.
On day 2, she writes the numbers 2 and 3.
On day 3, she writes the numbers 4, 5, and 6.
Paris continues writing numbers in this way, writing N numbers on the N th day.

- a. What is the largest number that Paris writes on the 20th day? Show or explain how you obtained your answer.

From the pattern, the largest number written on day N equals the sum of the first n natural numbers $= \frac{N(N+1)}{2}$. For instance, the largest number written on day 1 $= \frac{1(1+1)}{2} = 1$, the largest number written on day 2 $= \frac{2(2+1)}{2} = 3$, the largest number written on day 3 $= \frac{3(3+1)}{2} = 6$.

- c. What is the sum of the numbers that Paris writes on the N th day? Justify your answer.

4. a. Determine the positive integer c for which $\frac{1}{4} - \frac{1}{c} = \frac{1}{6}$

Multiply the equation by $12c$, so that $3c - 12 = 2c$. So, $c = 12$

- b. Determine the number of pairs of positive integers (d, e) for which $\frac{1}{d} - \frac{1}{e} = \frac{1}{12}$

Multiply the equation by $12de$ to obtain $12e - 12d = de$.

So:

$$0 = de + 12d - 12e$$

$$0 = d(e + 12) - 12e$$

$$-144 = d(e + 12) - 12e - 144$$

$$-144 = d(e + 12) - 12(e + 12) = (d - 12)(e + 12)$$

Since $e > 0$, $e + 12 > 0$ and $d - 12 < 0$ (since the product

| $d-12$ | $e+12$ | d | e |
|--------|--------|-----|-----|
| -1 | 144 | 11 | 132 |
| -2 | 72 | 10 | 60 |
| -3 | 48 | 9 | 36 |
| -4 | 36 | 8 | 24 |
| -6 | 24 | 6 | 12 |
| -8 | 18 | 4 | 6 |
| -9 | 16 | 3 | 4 |

For every prime number p , there are at least two pairs (r, s) of positive integers for which $\frac{1}{r} - \frac{1}{s} = \frac{1}{p^2}$

$T(4,1) = 4$. Note, the result is the same if the point is added on the boundary of an interior triangle.

If an interior point is added to one of the $T(4,1)$ triangles, $T(4,1)$ is reduced by one triangle and three new triangles are added (as below), with the net result of plus two triangles. So, $T(4,2) = 6$

Overall, the pattern is $T(4,k) = 2(k+1)$. **A proof by induction would provide additional points!**

So, $T(4,100) = 2(100 + 1) = 202$ triangles.

c. Determine the value of n for which $T(n,n) = 1000$

6. Define the function $f(x) = \frac{x}{x-1}$ for $x \neq 1$.

a. Determine all real numbers $r \neq 1$ for which $f(r) = r$

$$\text{If } f(r) = r, \text{ then } \frac{r}{r-1} = r$$

Since $r \neq 1$, $r = r(r-1)$ or $r = r^2 - r$.

Thus, $r^2 - 2r = r(r-2) = 0$, so $r = 0$ or $r = 2$.

b. ~~Determine all real numbers $x \neq 1$ for which $f(f(x)) = x$.~~

$$\text{Since } f(x) = \frac{x}{x-1}, f(f(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-(x-1)} = x$$

c. Suppose k is a real number and define $g(x) = \frac{2x}{x+k}$ for $x \neq -k$. Determine all real values of k for which $g(g(x)) = x$, where $g(x) \neq -k$.

If $x \neq -k$, then $g(x)$ is defined for all other real values.

If $x \neq -k$ and $g(x) \neq -k$, then $g(g(x)) = \frac{2\left(\frac{2x}{x+k}\right)}{\frac{2x}{x+k}+k}$ is defined for all other real values.

$$\text{Let } g(g(x)) = \frac{2\left(\frac{2x}{x+k}\right)}{\frac{2x}{x+k}+k} = x.$$

$$\text{Then, } \frac{4x}{2x+k(x+k)} = x$$

$$\begin{aligned} \text{So: } 4x &= x(2x + k(x+k)) \\ 4x &= x(2x + kx + k^2) \\ 4x &= 2x^2 + kx^2 + k^2x \\ 0 &= (2+k)x^2 + (k^2-4)x \\ 0 &= (k+2)[x^2 + (k-2)x] \end{aligned}$$

If $k = -2$, $\frac{4x}{2x+k(x+k)} = x$ for all values of x for which the function is defined (i.e., $x \neq -k$, and $g(x) \neq -k$)